What's the Modified DFT?

Keith Kumm

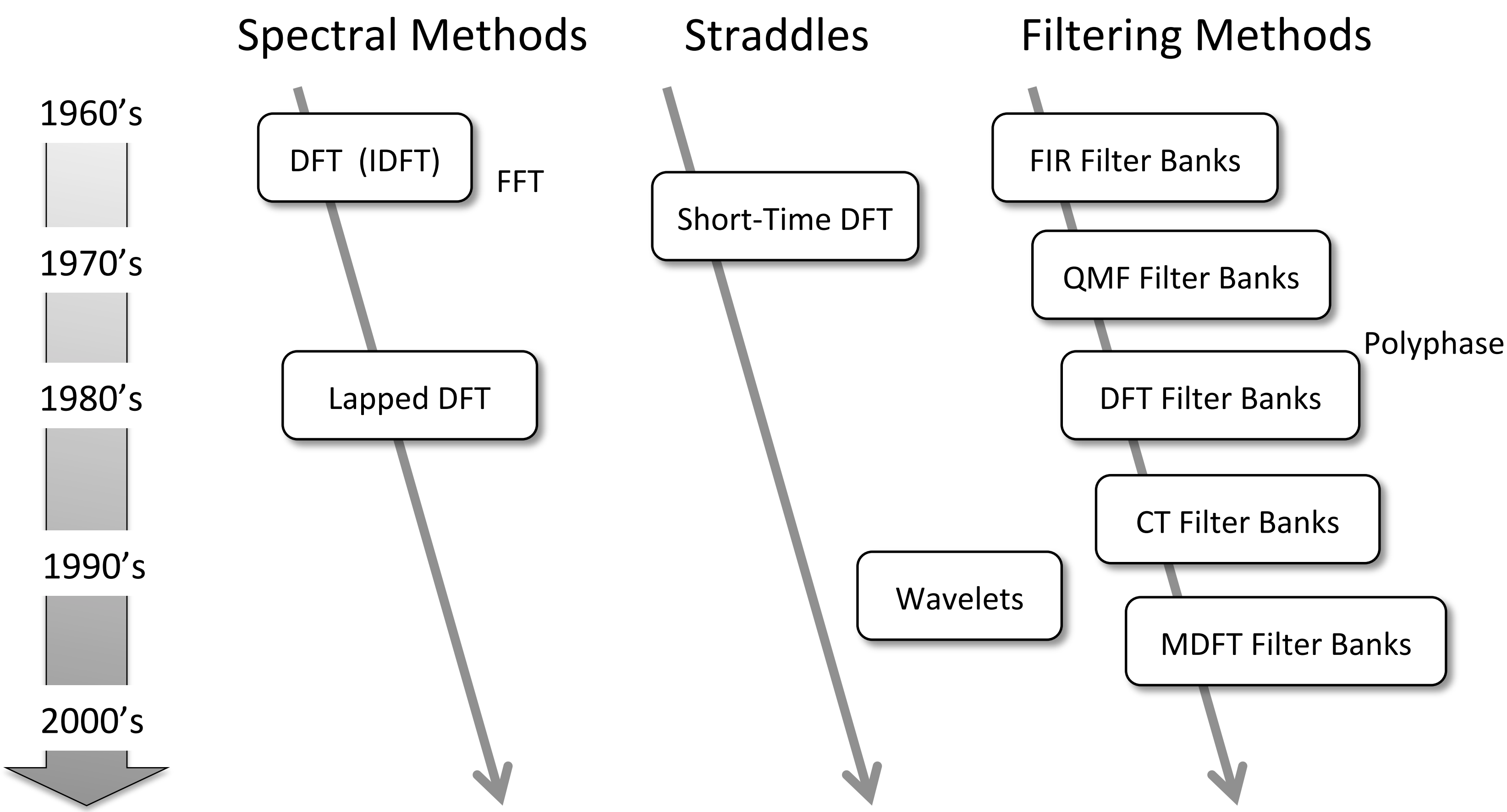
Short Talk Topic

Coverage

- Goal is for Everyone to Understand MDFT Basics
- Background Comments on Filter Banks in general
- Prime Motive: Perfect Reconstruction
- Warm-up: A few fundamental relations & memories
- QMF, a Two-Channel FB
- Polyphase Decomposition with DFT Matrix
- Uniform DFT, an M-Channel FB
- MDFT (finally!)
- Some Applications

A “Natural History” of Analysis (Synthesis)

By 1990, Plenty of Clues Pointed to the MDFT



MDFT Evolved from DFT Filter Banks

- **Grail: Perfect Reconstruction (PR) with (High) Resolution & Efficiency**

$$\hat{X}(z) = cz^{-k} X(z)$$

  - Meaningful if Reconstruction is Part of the Design Objective
  - Often, Modified Reconstruction is the Actual Objective
  - *Also* Leads to High Resolution, High Accuracy Analysis (or Synthesis)
  - Non- or Near-PR means

$$\hat{X}(z) = cz^{-k} X(z) \quad \hat{X}(z) = cz^{-k} T(z) X(z) + \text{aliasing terms}$$

- Way Forward
  - Obtain Filter Passband Flat Amplitude and Linear Phase
  - Obtain Passband Cross-over Equalization
  - Eliminate or Make Negligible Inter-Passband Aliasing
  - Exploit Complexity Reduction Inherent in Polyphase Realization

Warm-Up 1

- z-Transform
  - If convergent on unit circle

$$X\left(e^{j\omega}\right)=\sum_{k=-\infty}^{\infty}x\left(k\right)e^{-j\omega k} \quad z=e^{j\omega} \quad X\left(z\right)=\sum_{k=-\infty}^{\infty}x\left(k\right)z^{-k}$$

- Delay (by n) $y\left(n\right)=x(n-n_0) \quad Y\left(z\right)=\sum_{k=-\infty}^{\infty}x\left(k-n_0\right)z^{-k}=z^{-n_0}\sum_{m=-\infty}^{\infty}x\left(m\right)z^{-m}$

- LTI Operations
  - Linear Time Invariant

$$y\left(n\right)=x\left(n\right)*h\left(n\right) \quad \leftrightarrow \quad Y\left(e^{j\omega}\right)=H\left(e^{j\omega}\right)X\left(e^{j\omega}\right)$$

$$y\left(n\right)=H\left(e^{j\omega_0}\right)e^{j\omega_0 n} \text{ if } x\left(n\right)=e^{j\omega_0 n}$$

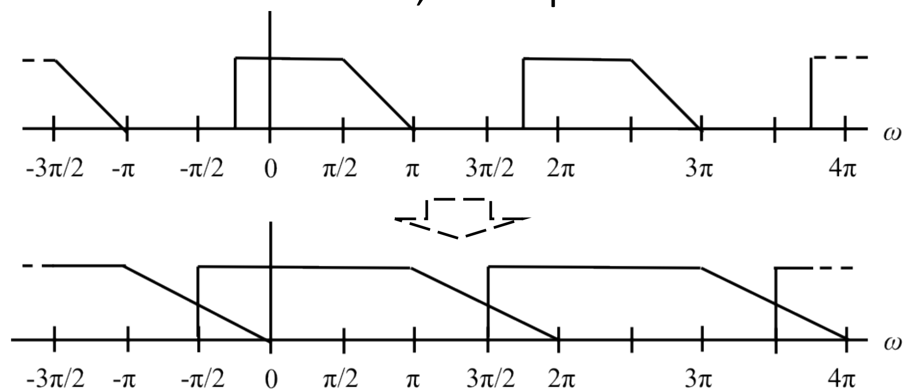
- LTV Operations
  - Linear Time Varying

Anything that behaves differently from sample to sample

Warm-Up 2

- Decimation (by M)   LTV !

“stretch & shift,” example for M = 2



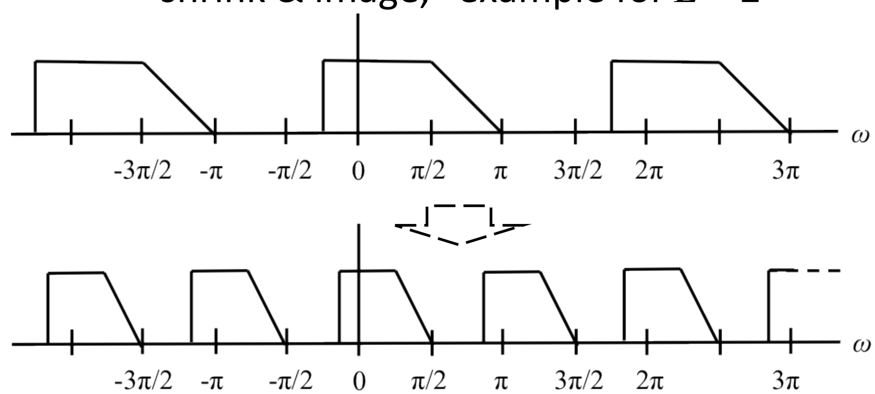
$$\Omega_s = 2\pi / T_s \longrightarrow \boxed{\downarrow M} \longrightarrow \Omega_s \to \Omega_s / M$$

(deleted samples)

$$Y_D\left(e^{j\omega}\right) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(e^{j(\omega-2\pi k)/M}\right)$$

- Expansion (by L)   LTV !

“shrink & image,” example for L = 2



$$\Omega_s = 2\pi / T_s \longrightarrow \boxed{\uparrow L} \longrightarrow \Omega_s \to L\Omega_s$$

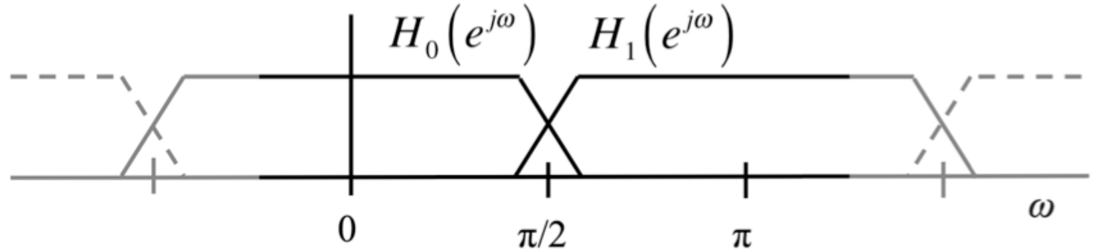
(interstitial 0’s)

$$Y_E\left(e^{j\omega}\right) = X\left(e^{j\omega L}\right)$$

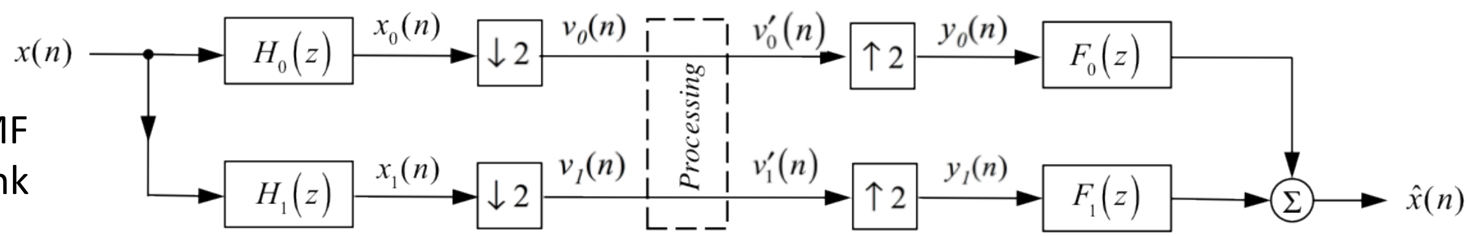
Quadrature Mirror Filter Bank (1970's)

- All x, v, y are time domain sequences
- $x(n), \hat{x}(n)$ are full bandwidth (e.g., for critical real sampling in $[0, \pi)$)

Analysis transfer functions, "critically" overlapped



QMF Bank



stretch/shift

aliasing happening

shrink/image

recover baseband and cancel aliases

reconstruct

QMF Alias Cancellation & PR

- By Careful Set-up of Synthesis Filters $F_0(z),\ F_1(z)$
- Works for Any Mirror-Symmetric $H_0(z),\ H_1(z)$ s.t. $H_1(z) = H_0^*(-z) = H_0(-z)$
- $\hat{X}(z) = F_0(z)Y_0(z) + F_1(z)Y_1(z)$

Now, Let Synthesis Filter Design be:

$$F_0(z) = H_1(-z),\ F_1(z) = -H_0(-z)$$

And via the Ref Dgm:

$$\hat{X}(z) = \frac{1}{2}\left[H_0(z)F_0(z) + H_1(z)F_1(z)\right]X(z) + \frac{1}{2}\left[H_0(-z)F_0(z) + H_1(-z)F_1(z)\right]X(-z)$$

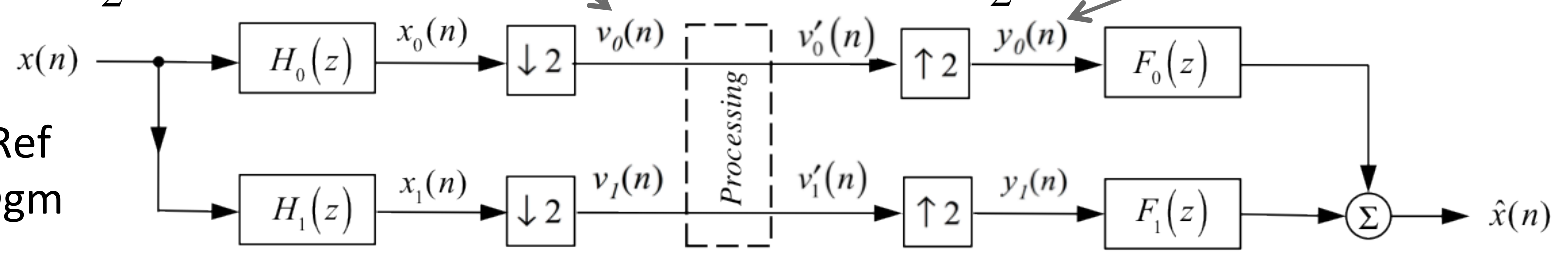
0 alias!

$$\hat{X}(z) = T(z)X(z) = \frac{1}{2}\left[H_0(z)H_0^*(z) + H_1(z)H_1^*(z)\right]X(z) = \frac{\left|H_{0+1}(z)\right|^2}{2} z^{-2h} X(z)$$

PR !

$$V_k(z) = \frac{1}{2}\left[X_k\left(z^{1/2}\right) + X_k\left(-z^{1/2}\right)\right] \qquad Y_k(z) = V_k\left(z^2\right) = \frac{1}{2}\left[H_k(z)X(z) + H_k(-z)X(-z)\right]$$

Ref Dgm



(Two Channel) Polyphase Decomposition

- Goal: eliminate redundancies, simplify computation
- "Modulation" matrix version of QMF *without* $\downarrow 2$ and $\uparrow 2$

$$\hat{X}(z) = \frac{1}{2}\begin{pmatrix} F_0(z) & F_1(z) \end{pmatrix}\begin{pmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{pmatrix}\begin{pmatrix} X(z) \\ X(-z) \end{pmatrix} = \frac{1}{2}\mathbf{F}_m(z)\mathbf{H}_m(z)\mathbf{x}_m(z)$$

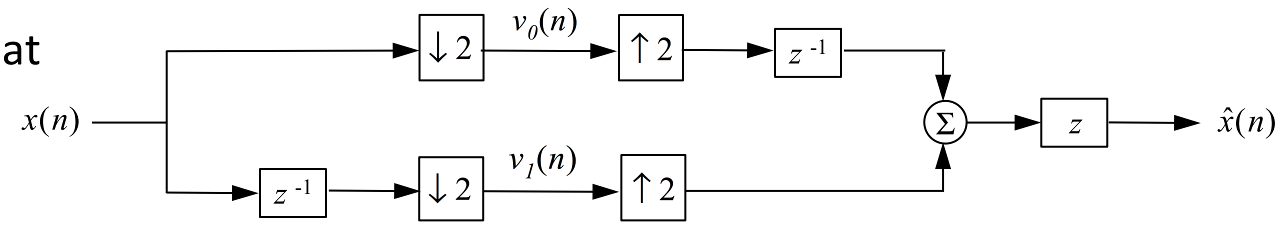
- Terms $H_k(z),\ H_k(-z)$ have same coefficients and are redundant
- If we can eliminate the multiplication redundancies*, we are ahead

  * even at the cost of some extra additions

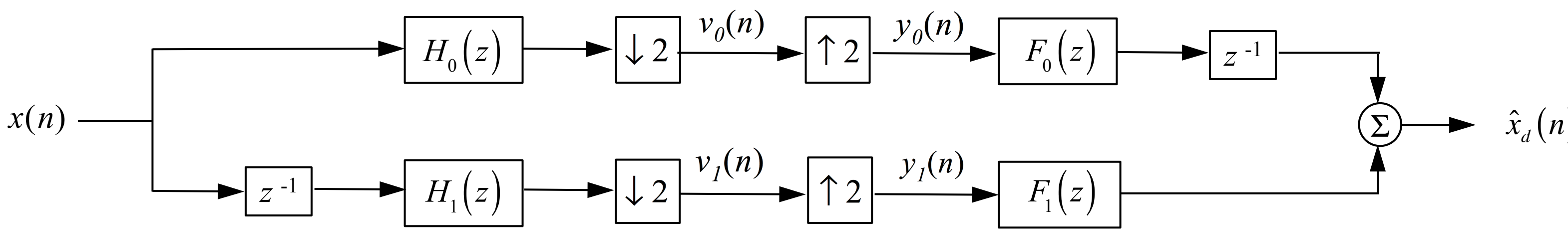
- Very similar to the computation "gain" of the FFT over the DFT

Polyphase QMF, 1

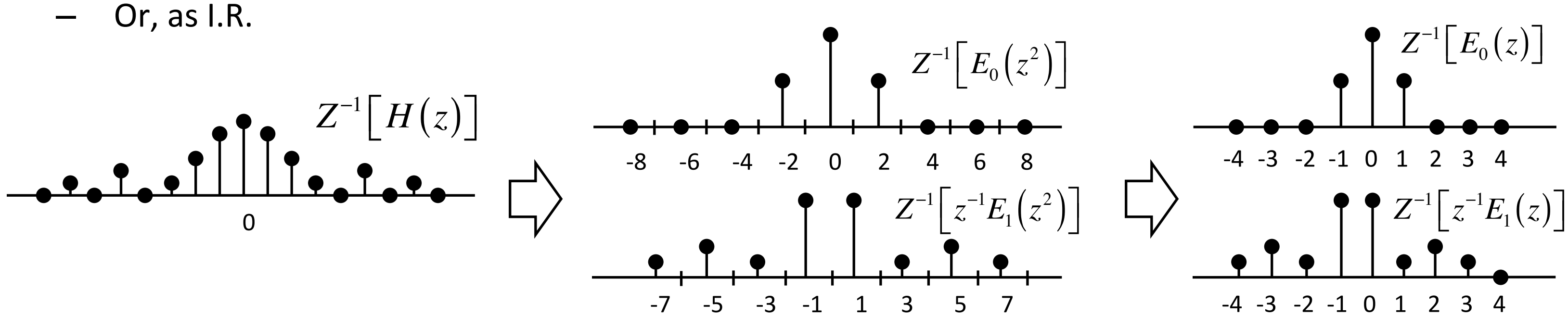
- Observe that



- Now Add Some Utility

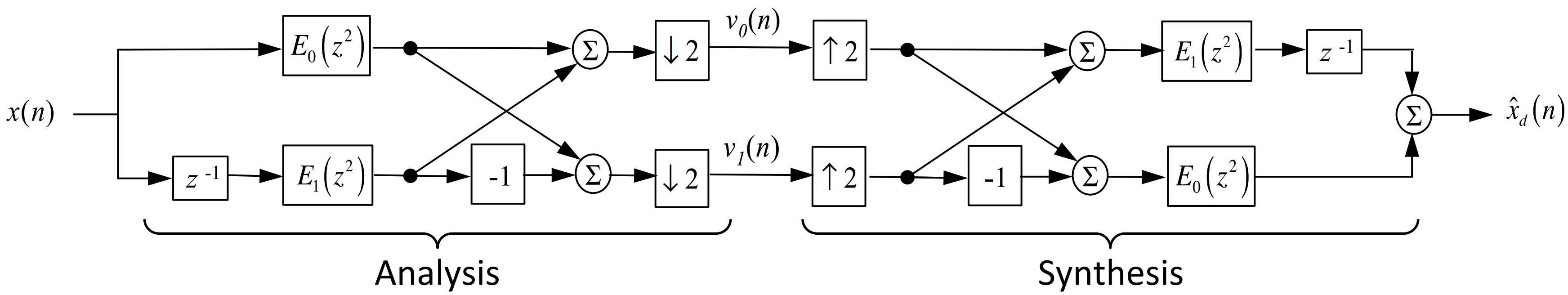


- Consider for H, an FIR $$H(z) = E_{even}\left(z^2\right) + z^{-1}\ E_{odd}\left(z^2\right) \equiv E_0\left(z^2\right) + z^{-1}\ E_1\left(z^2\right)$$
  - Or, as I.R.

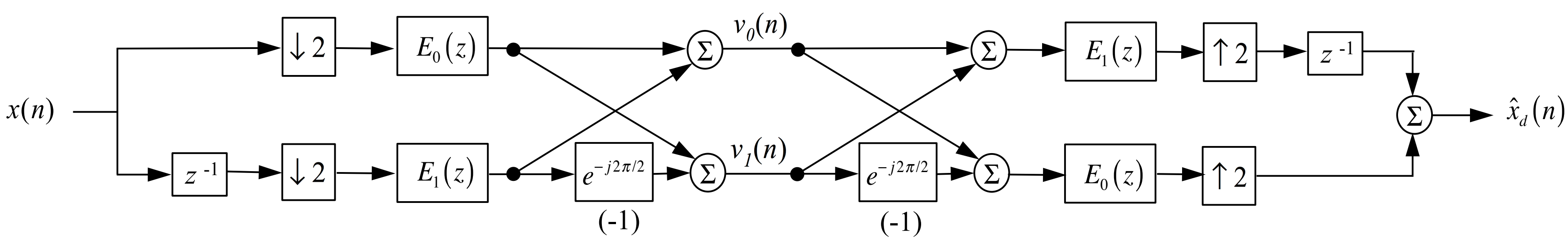


Polyphase QMF, 2

- Now Recall for QMF $H_1(z) = H_0(-z)$ $F_0(z) = H_1(-z),\ F_1(z) = -H_0(-z)$
- So on "Analysis" Side $\begin{pmatrix} H_0(z) \\ H_1(z) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} E_0(z^2) \\ z^{-1} E_1(z^2) \end{pmatrix}$

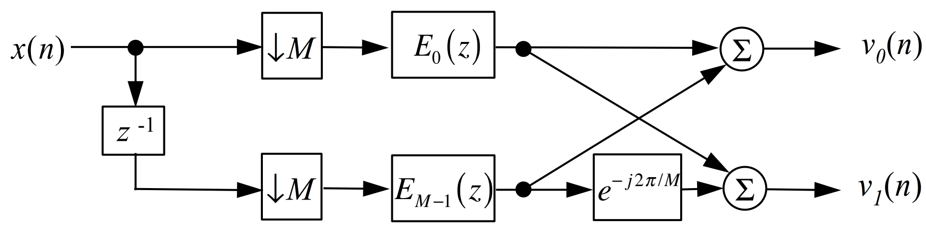


- Equivalently

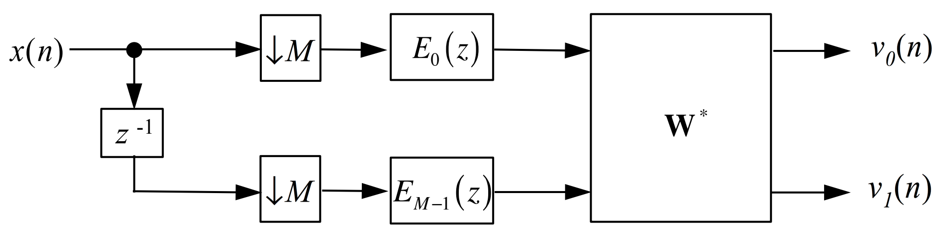


Polyphase Generalization, 1
with a Look-Ahead to DFT

- Consider Just Analysis Side of the QMF, where $M = 2$



- Turns Out that, again $M = 2$



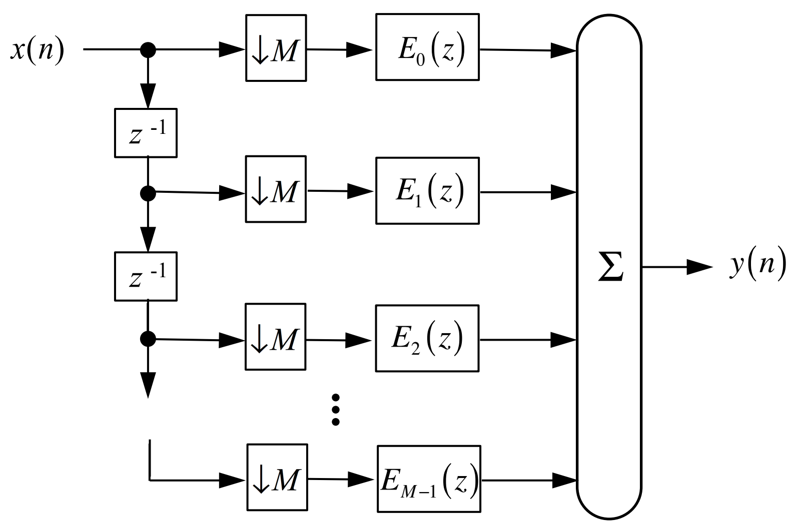
$$\mathbf{W}^* = \left[W_M^{-km} \right]$$

$$W_M = e^{-j2\pi/M}$$

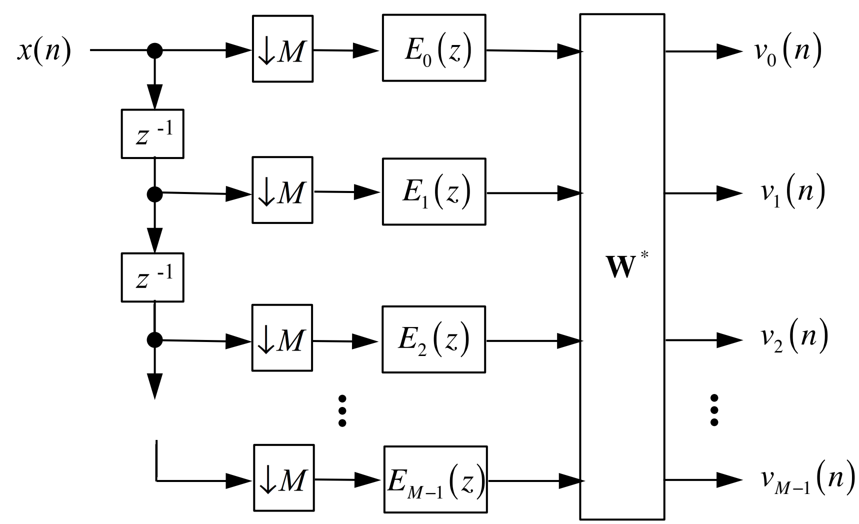
Polyphase Generalization, 2
with Sneak Peak at DFT Filter Bank

- We can Generalize to any M constrained only that $\log_2 M = K$, an integer

For a simple FIR



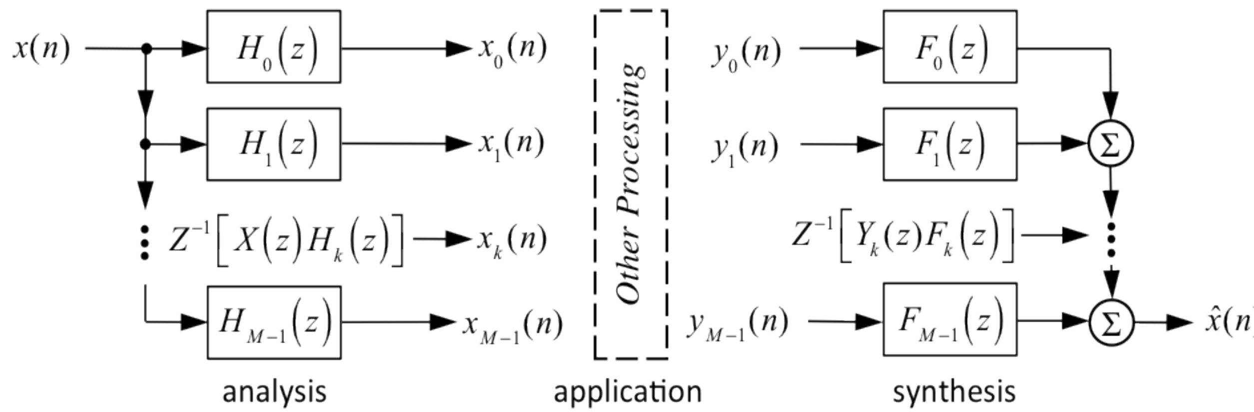
For a simple Filter Bank



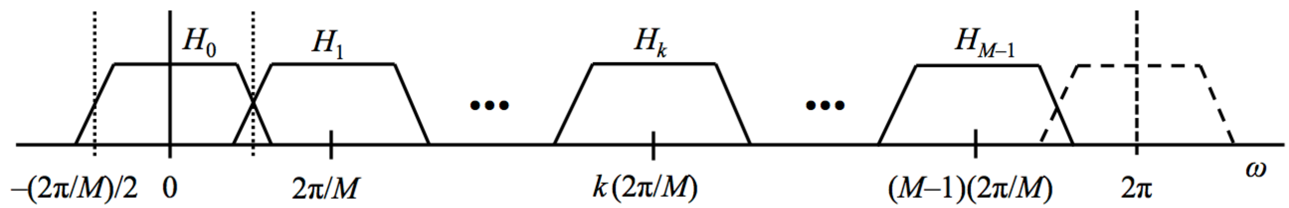
again $\mathbf{W}^* = \left[W_M^{-km}\right]$ $\quad W_M = e^{-j2\pi/M}$

General Discrete (Time) Filter Bank (1970's)

FIR type realization



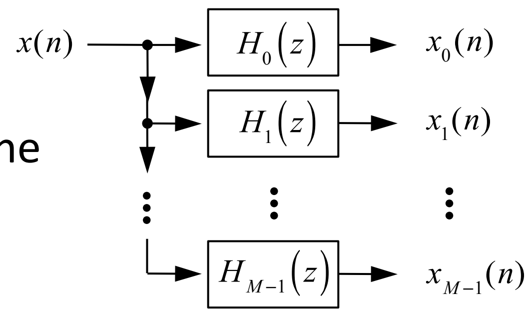
"critically" overlapped realization



- All x, y are time domain sequences
- $x(n), \hat{x}(n)$ are full bandwidth (e.g., for complex samples in $[0, 2\pi)$)
- $H_0(z)$, $F_0(z)$ are baseband in $[-(2\pi/M)/2, +(2\pi/M)/2)$
- $k = 1, M{-}1$: $H_k(z)$ are bandpass and downconvert, $F_k(z)$ upconvert
- $x_k(n),\ y_k(n)$ are all baseband processes centered on 0 rad
- Aliasing happens in the Processing, and also is not correctable at $\hat{x}(n)$

Discrete Filter Bank – Uniform Case

Consider just the Analysis side

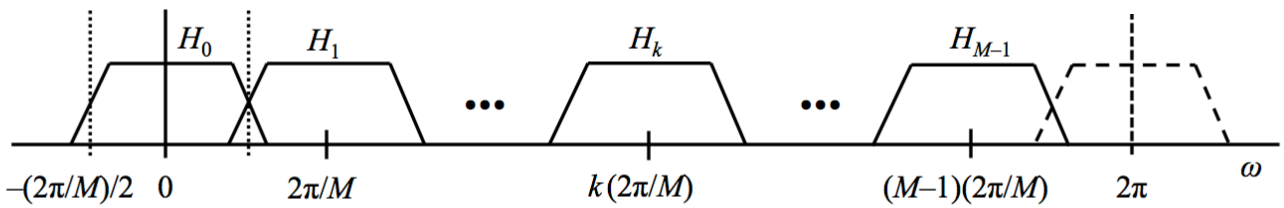


$$W = e^{-j2\pi/M}$$

$$H_0(z) = W^0\left(c_0 z^{-0} + c_1 z^{-1} + \ldots + c_{M-1} z^{-(M-1)}\right)$$

So for "Uniform" i.r.

$$H_0(z) = W^0\left(z^{-0} + z^{-1} + \ldots + z^{-(M-1)}\right) = \sum_{i=0}^{M-1} z^{-i}$$



Replica Filters on center freqs $\omega_k = (k/M)2\pi$ so $H_k(z) = H_0(zW^k)$

And for Uniform case $h_0(n) = W^0\delta(n-i) = \delta(n-i), \quad h_k(n) = W^{-ki}\delta(n-i)$

$H_0(z)$ will (eventually for MDFT) be called a "prototype" filter

$$X_k(z) = \sum_{i=0}^{M-1} H_k(z)X(z) = \sum_{i=0}^{M-1} H_0(zW^k)X(z) = \sum_{i=0}^{M-1}\left(\sum_{i=0}^{M-1}(zW^k)^{-i}\right)X(z) = \sum_{i=0}^{M-1} z^{-i}W^{-ki}X(z)$$

Discrete Fourier Transform (DFT) & Inverse DFT (IDFT)

N **x** N **DFT Matrix** $\mathbf{W}_N = \left[W_N^{km} \right], \quad W_{(N)} = W = e^{-j2\pi/N}$ symmetric !

Commonly employed as $X(k) = \sum_{m=0}^{N-1} W^{km} \, x(m)$

Observe that $\mathbf{W}^{\dagger}\mathbf{W} = N\,\mathbf{I}\,, \quad \mathbf{W}^{\dagger} = \left(\mathbf{W}^{T}\right)^{*} = \mathbf{W}^{*}$

Meaning that $\mathbf{W}^{-1} = \dfrac{\mathbf{W}^{\dagger}}{N} = \dfrac{\mathbf{W}^{*}}{N}$

$$\mathbf{W}_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & +j \\ 1 & -1 & 1 & -1 \\ 1 & +j & -1 & -j \end{pmatrix}$$

N **x** N **IDFT Matrix** $N^{-1}\,\mathbf{W}^{*} = N^{-1}\left[W_N^{-km} \right] = N^{-1}\left[\left(W_N^{km} \right)^{-1} \right]$

Commonly employed as $x(m) = \dfrac{1}{N} \sum_{m=0}^{N-1} W^{-km} \, X(k)$

or in matrices $\mathbf{x} = \mathbf{W}^{*}\,\mathbf{X}$

$$\mathbf{W}_4^{*} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & +j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & +j \end{pmatrix}$$

Applying IDFT to Uniform Discrete FB

Recall

Sticking with just the Analysis side

from IDFT (as a template)

$$x(m) = \frac{1}{N} \sum_{m=0}^{N-1} W^{-km} X(k)$$

$$\mathbf{x} = \mathbf{W}^* \mathbf{X}$$

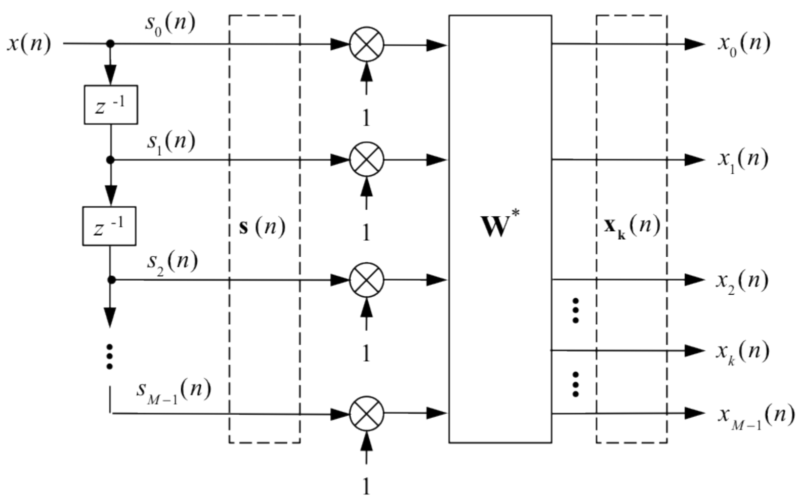
from Uniform Discrete FB

$$X_k(z) = \sum_{i=0}^{M-1} z^{-i} W^{-ki} X(z)$$

$$Z^{-1}\left[X_k(z)\right] = Z^{-1}\left[\sum_{i=0}^{M-1} W^{-ki} z^{-i} X(z)\right]$$

$$x_k(n) = \sum_{i=0}^{M-1} W^{-ki} x(n-i) = \sum_{i=0}^{M-1} W^{-ki} s_i(n)$$

$$\mathbf{x}_k = \mathbf{W}^* \mathbf{x}$$

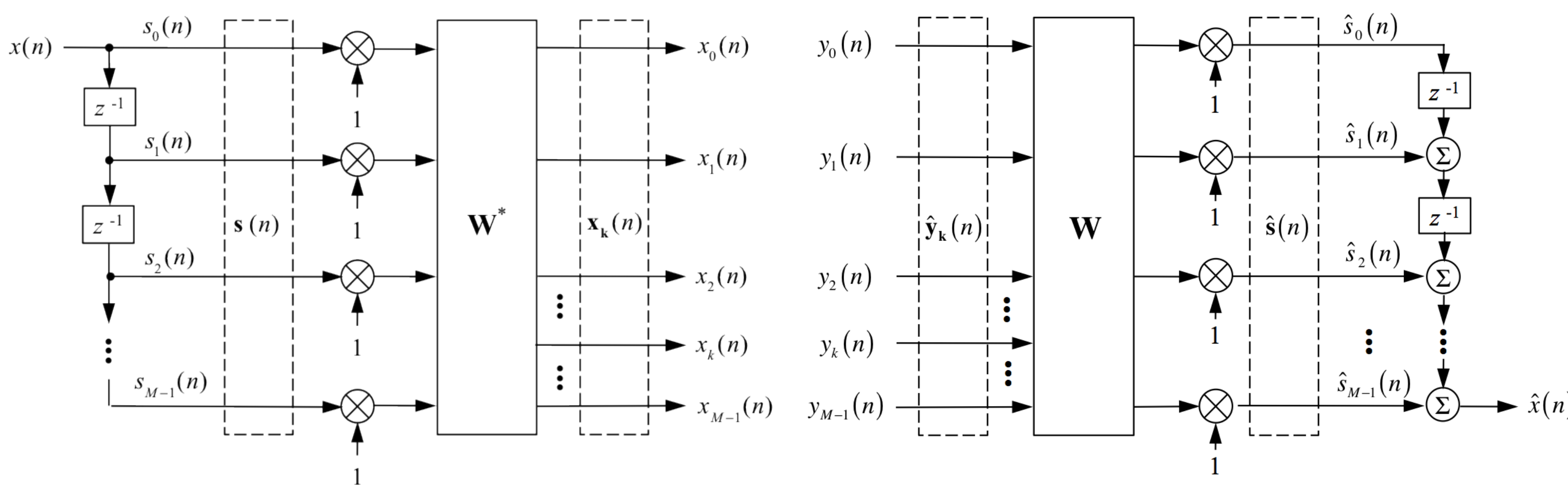


Uniform Complex-Modulated DFT FB (1980's)

A similar thing happens on Synthesis side, but this time involving the DFT Matrix $\mathbf{W}$

(since of course) $\mathbf{W}^{\dagger}\mathbf{W} = \mathbf{W}^{*}\mathbf{W} = (N/N)\mathbf{W}^{*}\mathbf{W} == N\ \mathbf{W}^{-1}\mathbf{W} = N\ \mathbf{I}$

And we end up with the C-M DFT Filter Bank



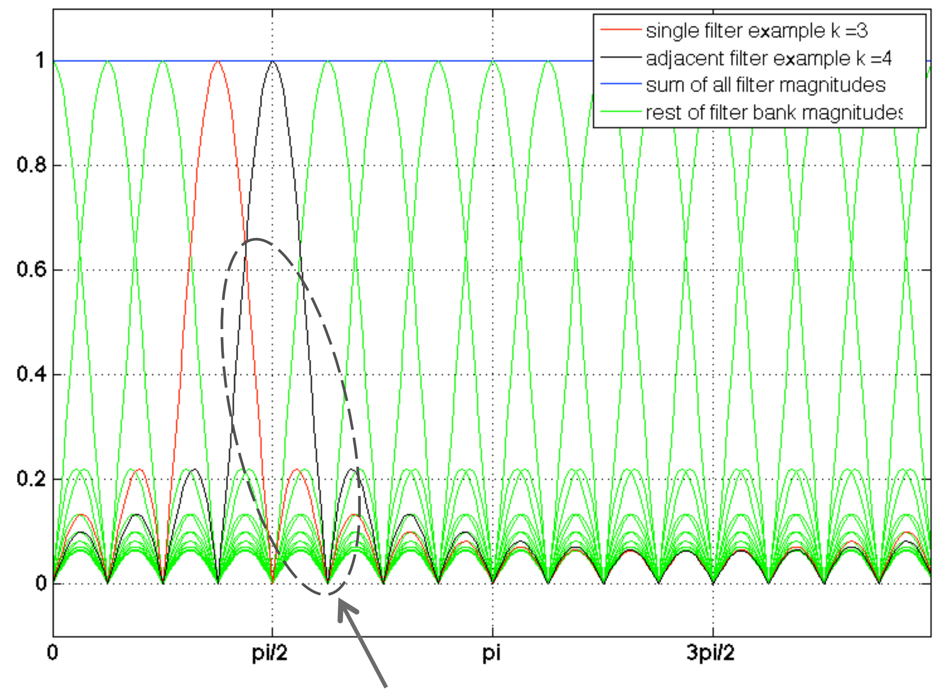
As Is, the Uniform C-M DFT Filter Bank is PR !

Uniform C-M DFT Analysis Freq Response

M = 16

$$\left|H_k\left(e^{j\omega}\right)\right|,\quad \left|H_3\left(e^{j\omega}\right)\right|,\quad \sum_{k=0}^{15}\left|H_k\left(e^{j\omega}\right)\right|$$

- Sum of individual filters (green & black) = 1 (blue) suggests PR !
- And indeed, entire bank (analysis + synthesis) can be made so
- But a single filter response (e.g., red at k = 3) is **very poor** due to only using the uniform window
- Hardly flatness and selectivity of a good FIR, much sidelobe aliasing
- If only a better filter element could be had – a big suggestion – without giving up the relative simplicity of the DFT structure



Inter-filter aliasing is a very significant effect in the FB's "Processing" interior. This doesn't mean however that the entire bank can't be made PR !

Uniform DFT FB Freq Response
Is that |sinc| or |"sinq"| ?

- Freq Response of Block of 1's, i.e. $H_0(z) = \sum_{i=0}^{M-1} z^{-i}$

Recall $\sum_{i=0}^{M-1} q^i = \frac{(q^M - 1)}{q-1}$ So $H_0(q^{-1}) = \sum_{i=0}^{M-1} q^i = \frac{(q^M - 1)}{q-1}$

$$H_0(e^{j\omega}) = \frac{(e^{-j\omega M} - 1)}{(e^{-j\omega} - 1)} = \left(\frac{e^{-jM\omega/2}}{e^{-j\omega/2}}\right) \frac{(e^{-jM\omega/2} - e^{jM\omega/2})/(2j)}{(e^{-jM\omega/2} - e^{jM\omega/2})/(2j)} = e^{-j(M-1)\omega/2} \frac{\sin(M\omega/2)}{\sin(\omega/2)}$$

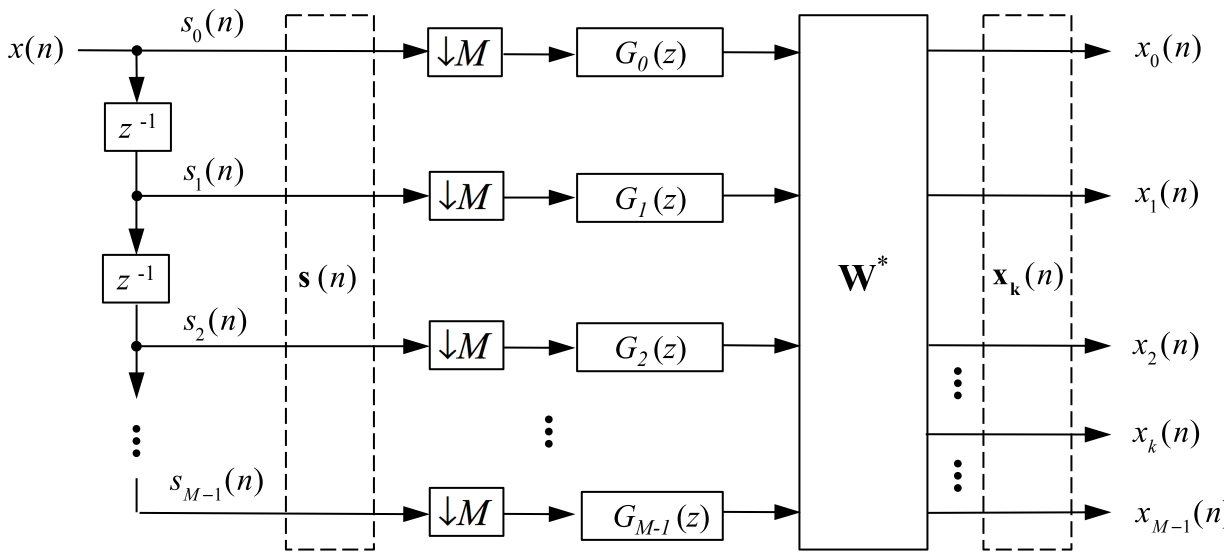
$$\left|H_0(e^{j\omega})\right| = \left|e^{j(M-1)\omega/2} \frac{\sin(M\omega/2)}{\sin(\omega/2)}\right| = \left|\frac{\sin(M\omega/2)}{\sin(\omega/2)}\right|$$

that's no sinc !
Call it a "sinq"

$$\angle H_0(e^{j\omega}) = \angle e^{j(M-1)\omega/2} + 0 = (M-1)\omega/2$$

General Complex-Modulated DFT FB with Polyphase Filtering, Analysis Side

Add a Better Filter, But There's a Catch



General C-M DFT Filter Bank is the Most Computationally Efficient FB Structure

But, General C-M DFT Filter Bank is No Longer PR !

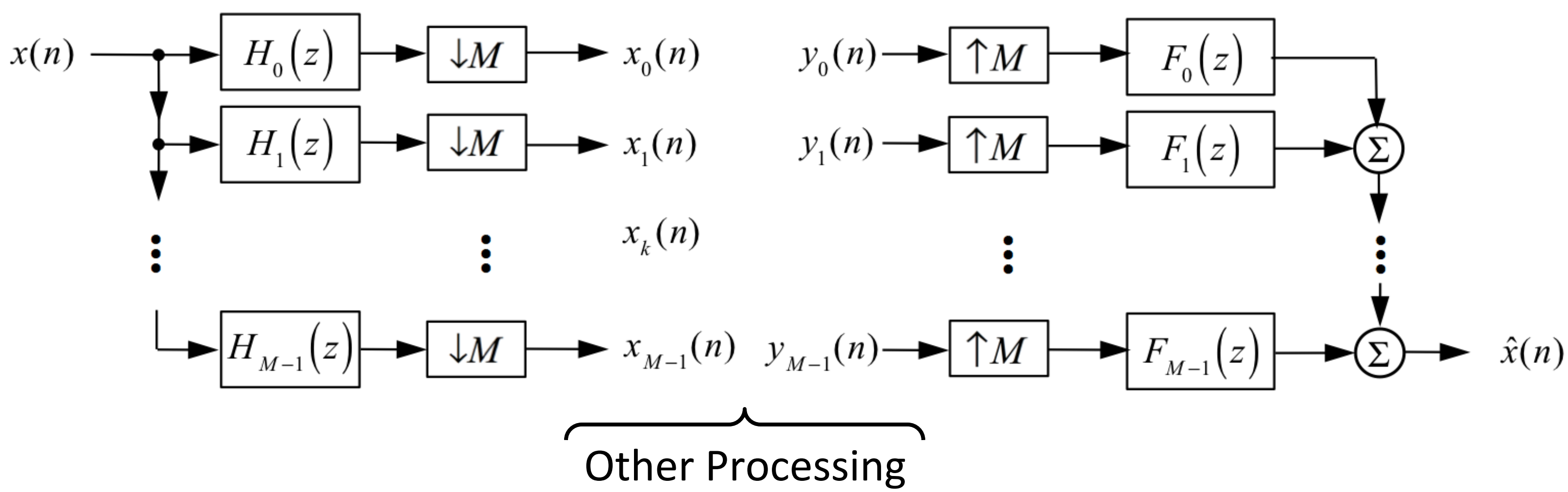
Credit

- Essential to Acknowledge Karp & Fliege at this Point
  - See Bibliography
  - As Far as I Know, They Discovered the MDFT
  - Most Following Diagrams and Observations follow Theirs
  - Some Streamlining and Focusing in This Short Topic Talk
  - There is NO NEW ORIGINAL WORK here
- We are Left to Exploit this Technology Wisely
  - No Advice is Offered on Patents and IP in MDFT

Modified DFT (MDFT)

(Not Yet Polyphase)

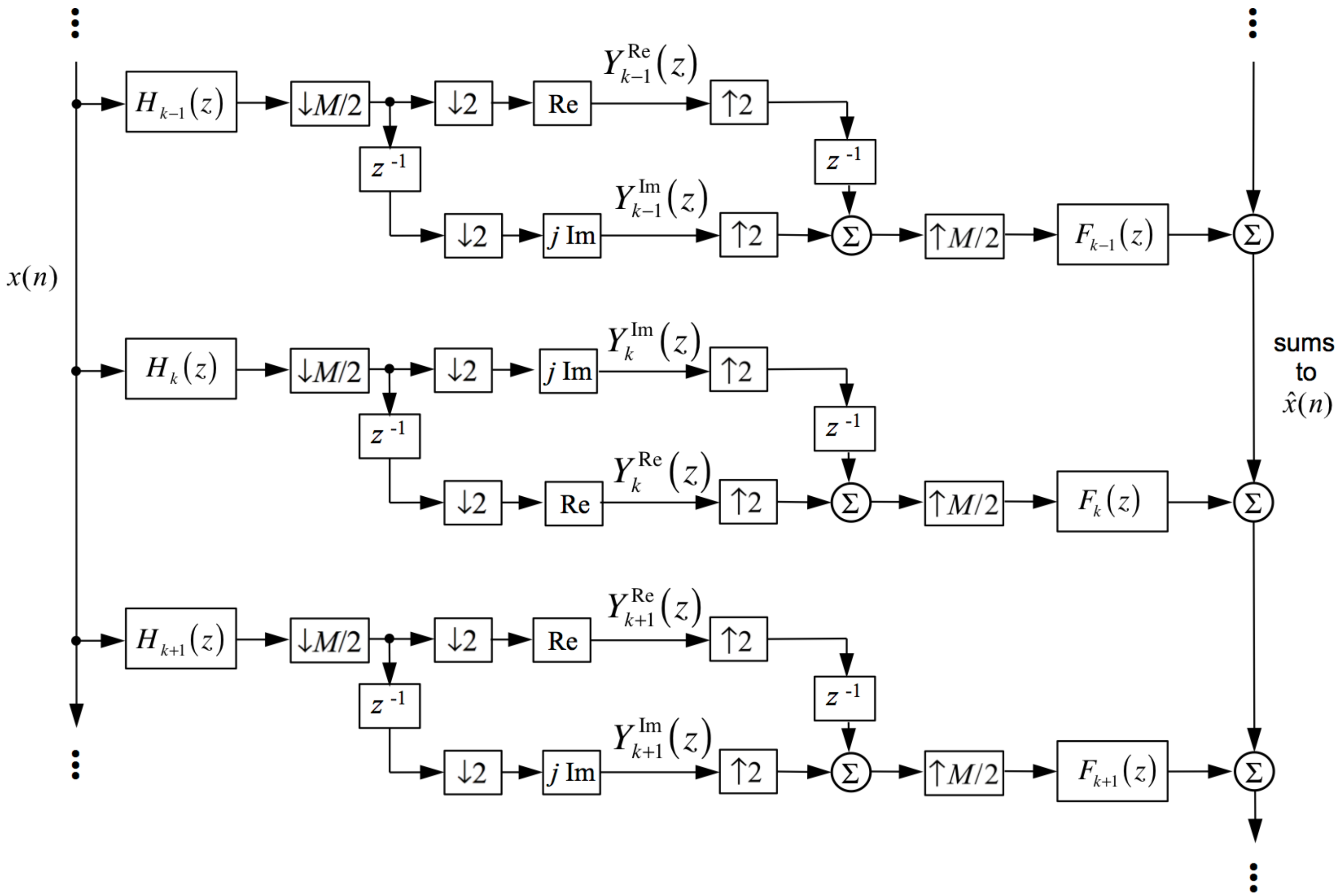
Start with our old friend the Discrete Filter Bank,
Add decimation (then expansion) to obtain high
computational efficiency for any Other Processing,
Then take the plunge into Alias Cancellation



Modified DFT (MDFT)

A Most Talented Stroke (Karp & Fliege, ~1993)

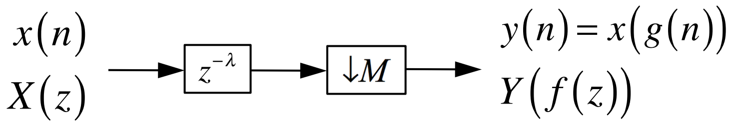
redraw Discrete FB this way, *almost* exact



Modified DFT (MDFT)

Supporting Concept

z-transform of a *delayed*, decimated signal



On the delay side*

$$y(n) = x(n \cdot M - \lambda),\ \lambda \geq 0$$

$$z^{-1} \leftrightarrow e^{1\cdot(+j2\pi/M)} = W^{-1},\ W = e^{-j2\pi/M}$$

$$z^{-\lambda} \leftrightarrow e^{\lambda\cdot(+j2\pi/M)} = W^{-\lambda}$$

* including $\downarrow M$ effect

On the decimation side

$$Y_D\left(e^{j\omega}\right) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(e^{j(\omega-2\pi k)/M}\right) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(e^{j\omega/M}W^k\right)$$

$$Y_D(z) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(z^{1/M}W^k\right),\ z = e^{j\omega}$$

$$Y_D\left(z^M\right) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(z^{M/M}W^k\right) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(zW^k\right)$$

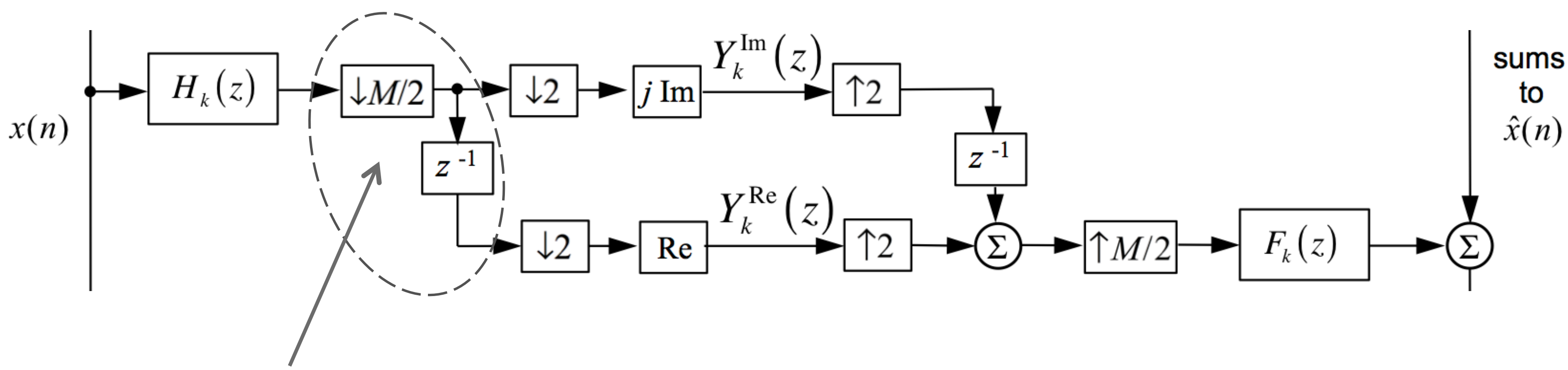
Put them together

$$z^{-\lambda}\, Y_D\left(z^M\right) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(zW^{k(1-\lambda)}\right) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(zW^k\right)W^{-\lambda k}$$

Modified DFT (MDFT)

Path to "Structure-based" Anti-Alias Cancellation

new structure gets some redundancy with a Twist



Due to decimation followed by delay, the lower path always gets $W_M^{-kM/2} = e^{jk\pi}$ extra phase in the l th lobe of the frequency response of $H_k(z)$

For l even, this is just 1, but for l odd, it is $\pm j$, alternating with k

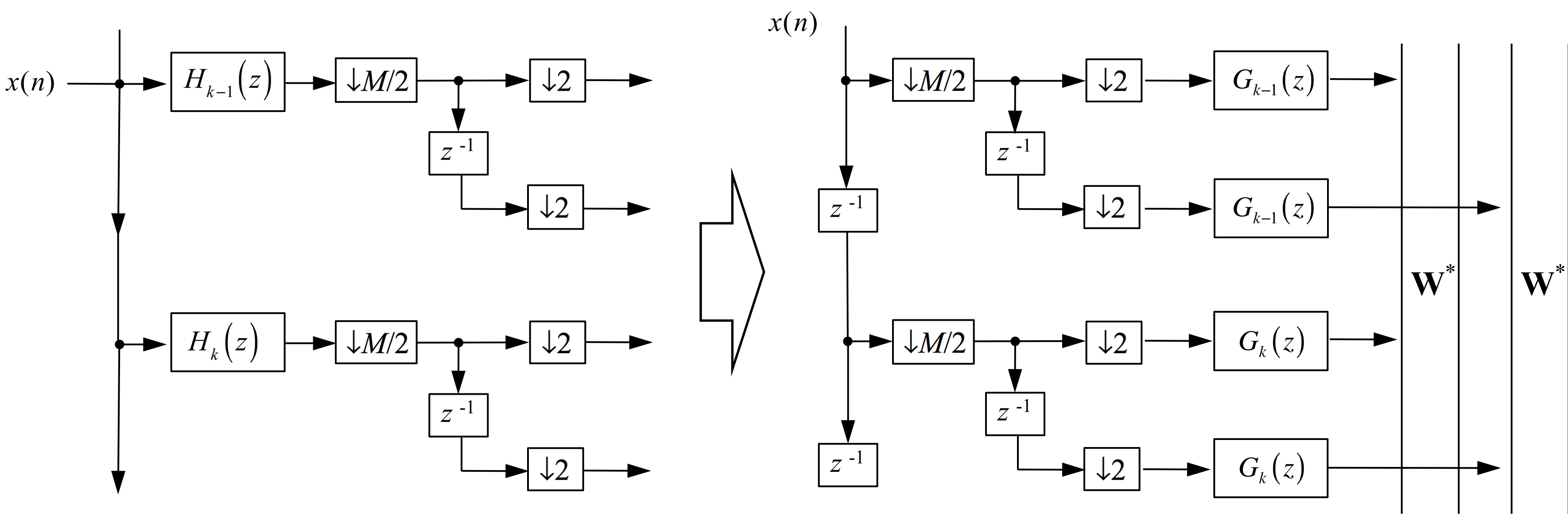
It is this very characteristic that forces odd sidelobe cancellation in the MDFT !

MDFT Filter Bank is "Almost Perfect" Reconstruction !

Modified DFT (MDFT)

Adding in Polyphase Efficiency, Step 1

using same transformation as for General DFT FB, Focus on Analysis Side



$G_k(z)$ is $1/M$ the length of $H_k(z)$

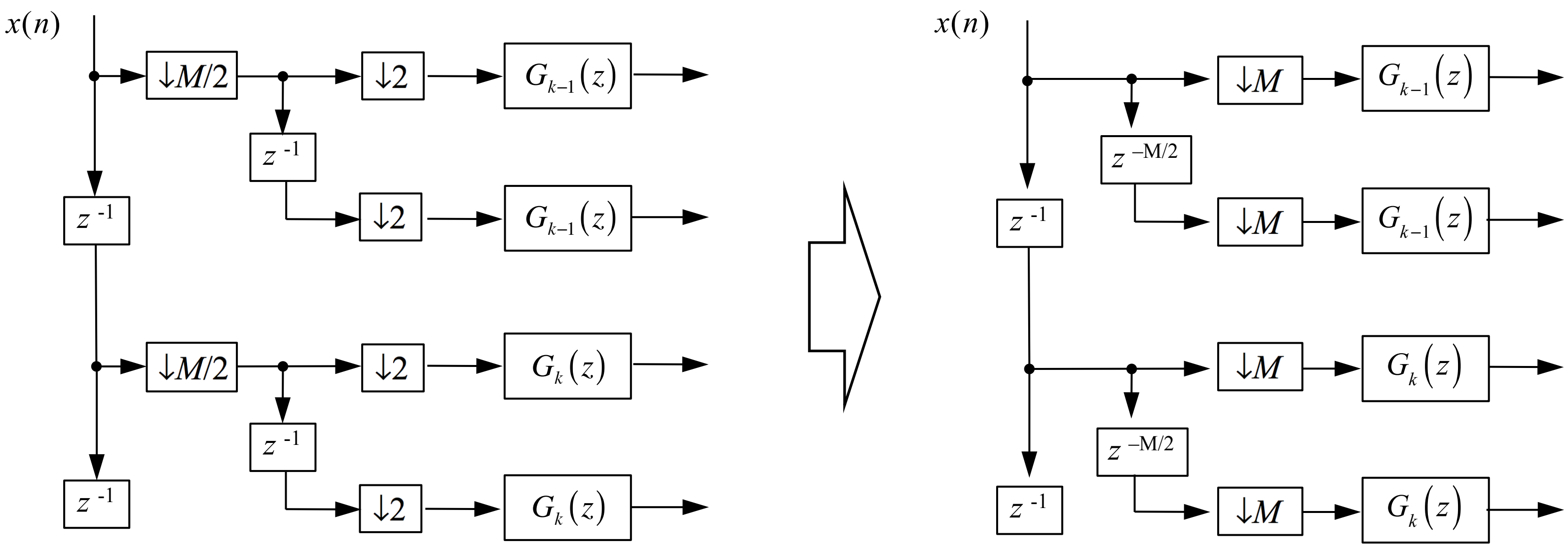
Polyphase MDFT Filter Bank will Achieve Both Highest Efficiency and Near PR !

Modified DFT (MDFT)

Intermediate Step 2

"pull through" the Unit Delay into an $M/2$ Delay, Analysis Side

this transformation is exact

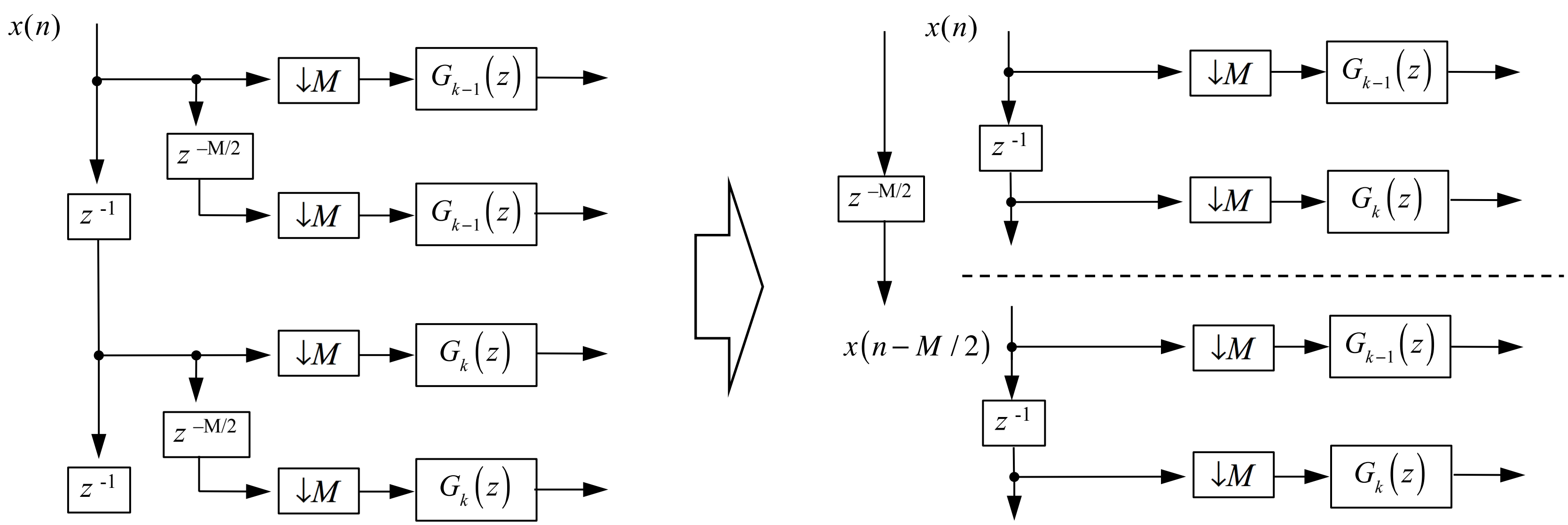


working toward a more regular repeated structure

Modified DFT (MDFT)

Get Back to Minimum Memory, Step 3

“pull through” the entire $M/2$–delayed “Lower Path,” Analysis Side



only take the $M/2$ delay once

Full-Up MDFT, "Dual-DFT" Configuration

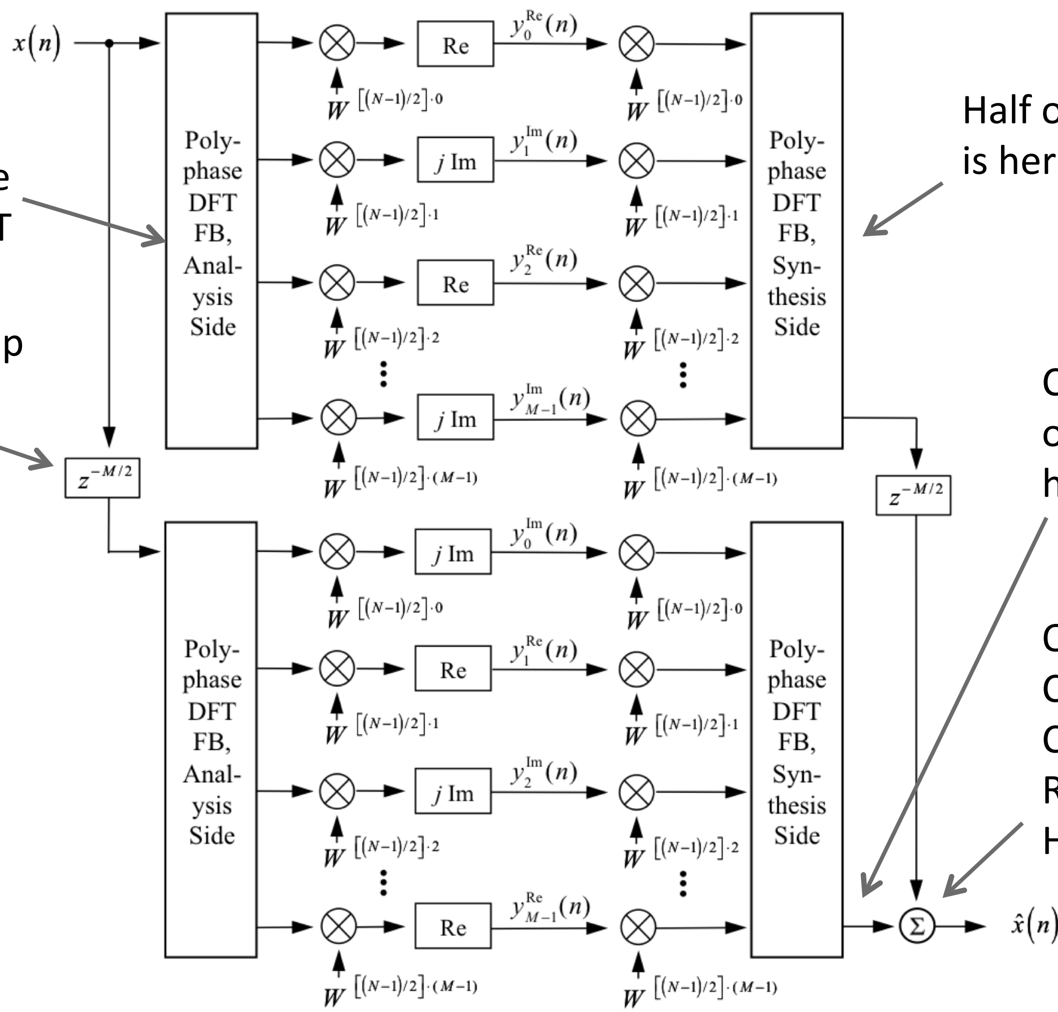
We've Seen How to Do the Polyphase DFT

The Magic Step to Odd-Order Alias Cancellation

Half of $x(n), X(z)$ is here

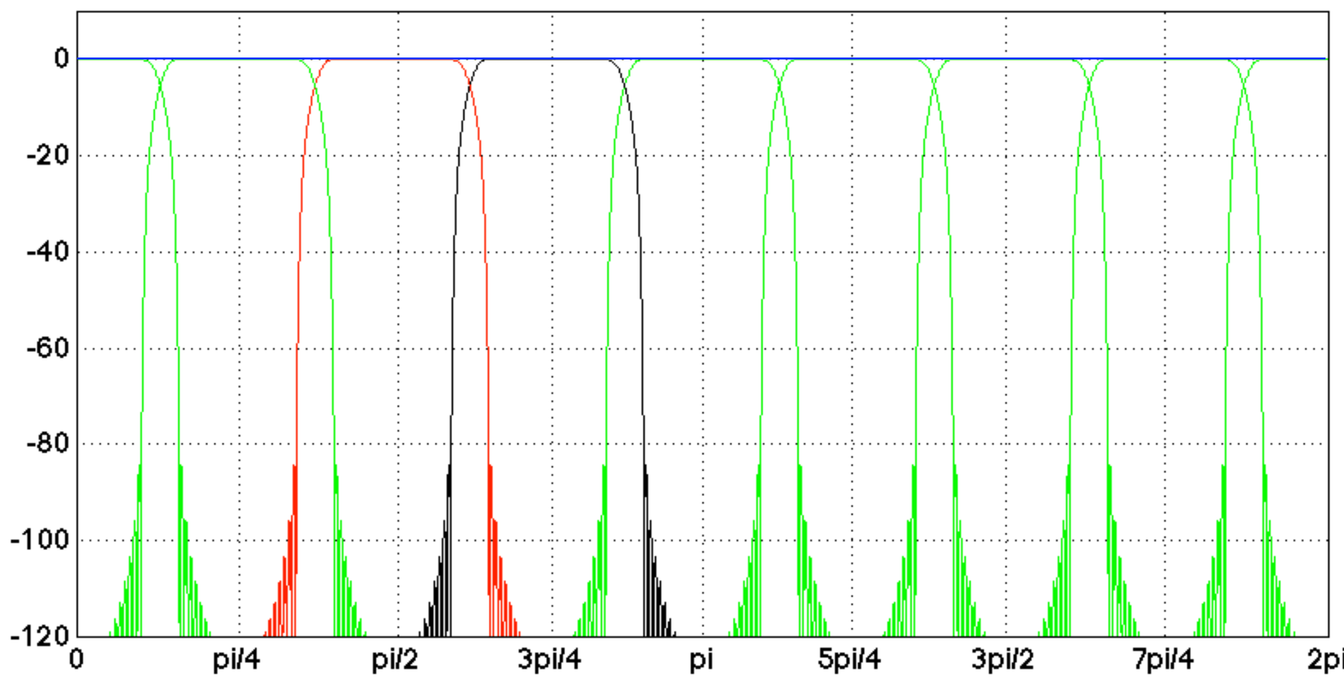
Complementary other half is here

Odd-Order Alias Cancellation *plus* Complex Sample Reconstruction Happens Here

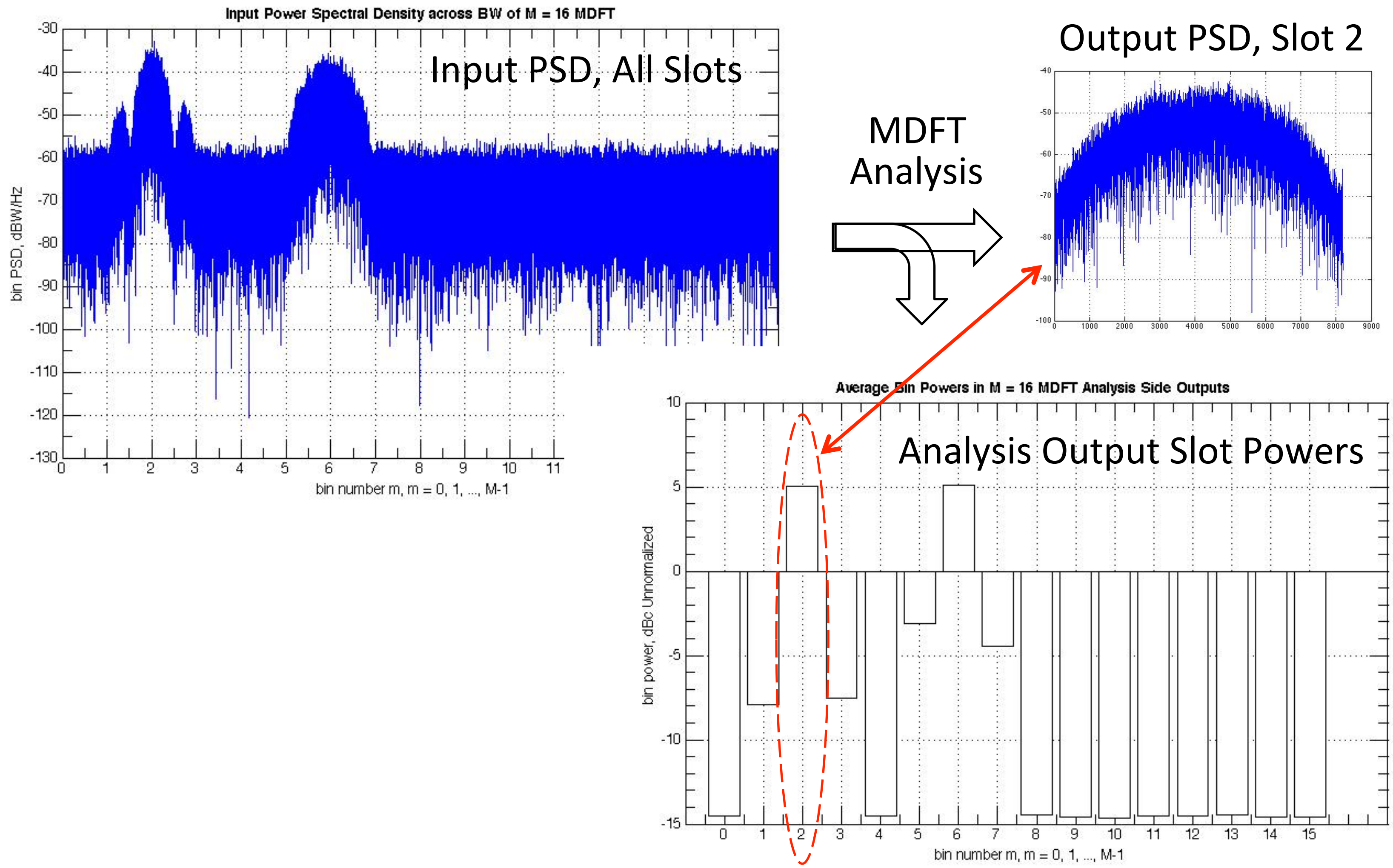


MDFT Analysis Side Freq Response

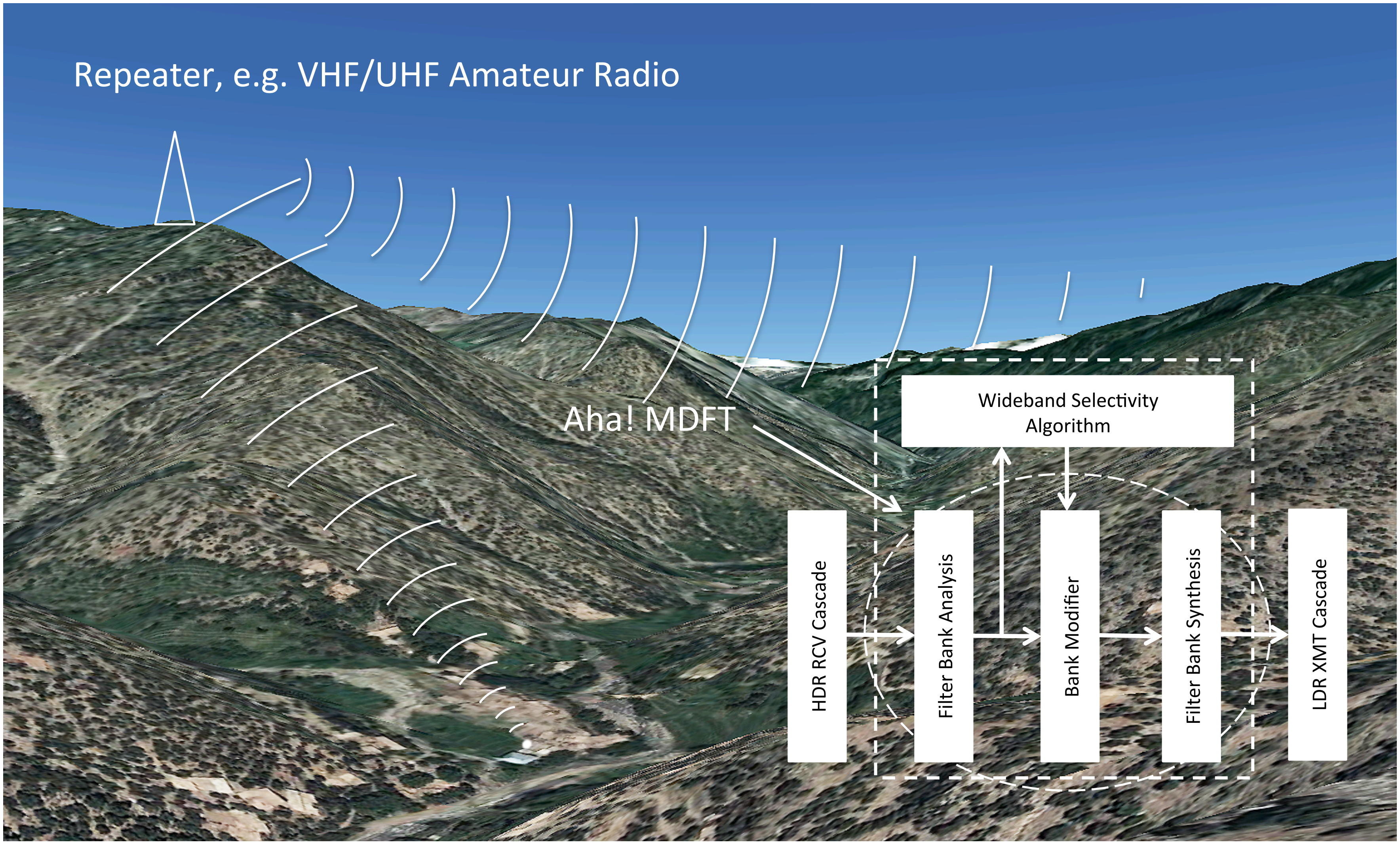
M = 8, α = 0.22 RRC Prototype Filter



Example, Analysis Side



An RF Type MDFT Application
When You See Everything, The Problem is Selectivity



Some Bibiography

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